

Dimensions of multi-fan algebras

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ABSTRACT. Given an arbitrary non-zero simplicial cycle and a generic vector coloring of its vertices, there is a way to produce a graded Poincare duality algebra associated to these data. The procedure relies on the theory of volume polynomials and multi-fans. This construction includes many important examples, such as cohomology of toric varieties and quasitoric manifolds, and Gorenstein algebras of triangulated homology manifolds, introduced by Novik and Swartz. In all these examples the dimensions of graded components of such duality algebras do not depend on the vector coloring. It was conjectured that the same holds for any simplicial cycle. We disprove this conjecture by showing that the colors of singular points of the cycle may affect the dimensions. However, the colors of smooth points are irrelevant. By using bistellar moves we show that the number of different dimension vectors arising on a given 3-dimensional pseudomanifold with isolated singularities is a topological invariant. This invariant is trivial on manifolds, but nontrivial in general.

1. Introduction

A multi-fan [8] is a collection of full-dimensional convex cones in the oriented space $V \cong \mathbb{R}^n$, emanating from the origin and equipped with weights. In contrast to the usual fans, cones in a multi-fan may overlap. The condition of completeness for a multi-fan is the following: the multi-fan Δ is complete if the weighted sum of its maximal cones is a cycle. This means that for every codimension one cone C the weights of all cones having C as a face sum to zero, if we count them with different signs depending on which side of C they are located. A multi-polytope based on a multi-fan Δ is a collection of affine hyperplanes each of which is orthogonal to some ray of Δ and intersecting whenever the corresponding rays are faces of some cone of Δ . In this work as well as in [1] we restrict to the situation when all cones of a multi-fan Δ are simplicial: in this case Δ is called simplicial multi-fan and corresponding multi-polytopes are called simple. In the simplicial case we use the following definitions

2010 *Mathematics Subject Classification.* Primary 05E40, 05E45, 28A75, 57M27; Secondary 13H10, 13P20, 13D40, 52C35, 52B70, 57N65, 13A02, 57M25, 05C15.

Key words and phrases. Volume polynomial, multi-fan, multi-polytope, invariants of 3-dimensional pseudomanifolds, vector coloring, Poincare duality algebra, Macaulay duality, bistellar moves.

This work is supported by the Russian Science Foundation under grant 14-11-00414.

DEFINITION 1.1 ([8, 1]). A *complete simplicial multi-fan* is a pair (ω, λ) , where

$$\omega = \sum_{I \subset [m], |I|=n} w(I)I \in Z_{n-1}(\Delta_{[m]}^{(n-1)})$$

is a simplicial cycle on m vertices, and $\lambda: [m] \rightarrow V$ is any function satisfying the condition: $\{\lambda(i) \mid i \in I\}$ is a basis of V if $|I| = n$ and $w(I) \neq 0$. In this case λ is called a *characteristic function*. If ω is the fundamental cycle of $(n-1)$ -dimensional oriented pseudomanifold K , we say that Δ is supported on K .

A *simple multi-polytope* P is a pair $(\Delta, \{H_1, \dots, H_m\})$, where Δ is a simplicial multi-fan, and H_i is a hyperplane in V^* orthogonal to $\lambda(i) \in V$, that is

$$H_i = \{x \in V^* \mid \langle x, \lambda(i) \rangle = c_i\}.$$

We say that P is based on Δ . The numbers $c_1, \dots, c_m \in \mathbb{R}$ are called *the support parameters* of a multi-polytope P .

Every simple multi-polytope based on a given simplicial multi-fan is uniquely characterized by the set of its support parameters.

Many common notions and facts about convex polytopes and fans are naturally extended to multi-fans and multi-polytopes. In particular, whenever P is a multi-polytope based on a complete multi-fan, there is a well-defined notion of volume of P , see [5]. For simple multi-polytopes there is a formula for volume, generalizing Lawrence formula for simple polytopes [1, 7]. Considering volumes of all multi-polytopes based on a fixed multi-fan Δ at once, we get the volume function $V_\Delta: \mathbb{R}^m \rightarrow \mathbb{R}$ defined on the vector space of support parameters. To a tuple $(c_1, \dots, c_m) \in \mathbb{R}^m$ this function associates the volume of the multi-polytope with support parameters (c_1, \dots, c_m) based on Δ . The function V_Δ is a homogeneous polynomial of degree n in the support parameters: $V_\Delta \in \mathbb{R}[c_1, \dots, c_m]_n$. It is called *the volume polynomial* of a multi-fan Δ .

There is a standard procedure to make a Poincare duality algebra out of any homogeneous polynomial, called Macaulay duality [9]. In the case of volume polynomials it was introduced and studied in [6, 12]. Consider the graded ring of polynomials $\mathcal{D} = \mathbb{R}[\partial_1, \dots, \partial_m]$ where each symbol ∂_i denotes $\frac{\partial}{\partial c_i}$, the partial derivative in i -th support parameter. For topological reasons we double the degree assuming $\deg \partial_i = 2$. Each variable ∂_i acts on $\mathbb{R}[c_1, \dots, c_m]$ in a natural way, so we may consider the homogeneous ideal $\text{Ann } V_\Delta = \{D \in \mathcal{D} \mid DV_\Delta = 0\}$ of differential operators annihilating V_Δ . The quotient algebra $\mathcal{A}^*(\Delta) \stackrel{\text{def}}{=} \mathcal{D} / \text{Ann } V_\Delta$ satisfies Poincare duality conditions: its top component $\mathcal{A}^{2n}(\Delta)$ is one-dimensional and the pairing $\mathcal{A}^{2j}(\Delta) \otimes \mathcal{A}^{2n-2j}(\Delta) \xrightarrow{\times} \mathcal{A}^{2n}(\Delta)$ is non-degenerate. We call $\mathcal{A}^*(\Delta)$ the duality algebra of a multi-fan. Let d_j denote the dimension of the graded component \mathcal{A}^{2j} . Poincare duality implies $d_j = d_{n-j}$.

Multi-fan algebras and the way they were constructed seem to be very important. When Δ is just the ordinary complete simplicial rational fan, the algebra $\mathcal{A}^*(\Delta)$ coincides with the cohomology of the corresponding toric variety (see [6]). Timorin [12] gave a purely geometrical proof of Stanley's g-theorem for a polytopal fan Δ by showing that $\mathcal{A}^*(\Delta)$ satisfies Lefschetz property. In [1] we proved that whenever the multi-fan Δ is supported

on an orientable homology manifold K , the algebra $\mathcal{A}^*(\Delta)$ coincides with the Gorenstein algebra introduced by Novik and Swartz in [10]. However, we also showed that every (finite-dimensional commutative) Poincare duality algebra generated in degree 2 is isomorphic to $\mathcal{A}^*(\Delta)$ for some multi-fan Δ . Therefore there exist complete simplicial multi-fans whose algebras do not satisfy Lefschetz property and whose dimensions' sequence (d_0, d_1, \dots, d_n) is not unimodal.

By Definition 1.1 a complete simplicial multi-fan Δ have two pieces of information: a simplicial cycle ω , the combinatorial data, and a characteristic function $\lambda: [m] \rightarrow V$, which encodes the directions of rays of a multi-fan, the geometrical data. In many cases the dimensions of graded components $d_j = \dim \mathcal{A}^{2j}(\Delta)$ depend only on ω , but not on λ . When Δ is supported on a sphere, the dimensions coincide with the h-numbers of this sphere. More generally, if Δ is supported on a manifold, the dimensions d_j coincide with the so called h"-numbers of a manifold, which are the combinatorial invariants of this manifold (see [10]). We suggested this was a general phenomenon [1, Conj.1].

PROBLEM 1.2. *Is it true that dimensions (d_0, d_1, \dots, d_n) of a multi-fan algebra $\mathcal{A}^*(\Delta)$ depend only on ω but not on characteristic function λ ? If no, is it true for multi-fans supported on pseudomanifolds?*

In this paper we answer both questions in negative by providing two counter-examples. First counter-example is computed by hand and relies on several simple facts proved previously in [1]. However, this counter-example is not a pseudomanifold so it does not answer the second question. Second question is more complicated since, in a sense, the easiest interesting example of a pseudomanifold which is not a manifold is the suspension over a 2-torus. In this case the volume polynomial is a polynomial of degree 4, and calculations can hardly be made by hand. However a simple program written in GAP [4, 3] allowed to answer the second question. If K is the suspension over the minimal triangulation of a torus, then there exist two multi-fans supported on K having d-vectors $(1, 5, 8, 5, 1)$ and $(1, 5, 12, 5, 1)$. Suspension over all other triangulations of a 2-torus, as well as suspensions over other orientable surfaces, are also discussed in the paper.

In a more theoretical part of this work we explain what makes manifolds so special from the view point of multi-fans. Let ω be a simplicial cycle and $i \in [m]$ be its vertex. The link of the vertex i in a cycle ω may be defined in a natural way. We say that i is a smooth vertex of a cycle ω if the link of i is (the fundamental cycle of) a homology sphere.

THEOREM 1.3. *Dimensions d_j of multi-fan algebra does not depend on the values of characteristic function in smooth vertices.*

In particular, since all vertices of a triangulated manifold are smooth, d-vectors of multi-fans supported on manifolds do not depend on characteristic function at all. Nevertheless, our examples show that d-vectors of mutli-fans may crucially depend on the values of λ in singular points.

It is natural to consider the following invariant of a simplicial pseudomanifold K : the number $r(K)$ of distinct dimension vectors of multi-fans supported on K . By the preceding discussion, this invariant equals 1 on homology manifolds, but it is nontrivial in general.

Using bistellar moves we show that this number is a topological invariant on the class of 3-dimensional pseudomanifolds with isolated singularities. The properties of this invariant are yet unknown.

The following family of examples is considered: take an arbitrary link $l: \bigsqcup_{\alpha} S_{\alpha}^1 \hookrightarrow S^3$ and collapse each of its components to a point. We conjecture that the resulting pseudo-manifold K has the property $r(K) = 1$ whenever all components of the link are pairwise unlinked. We checked that $r(K) = 1$ for disjoint union of knots and for Borromean rings. For Hopf link we have $r(K) = 2$ (this is just the case of suspension over a 2-torus).

Our considerations suggest that the additive structure of multi-fan algebras over pseudomanifolds may lead to new invariants of 3-pseudomanifolds or, at least, uncover interesting connections between convex geometry and 3-dimensional topology.

2. Necessary notions and facts

Let $\Psi \in \mathbb{R}[c_1, \dots, c_m]_n$ be an arbitrary non-zero homogeneous polynomial of degree n and let $\mathcal{D}^*/\text{Ann } \Psi$ be the corresponding Poincare duality algebra (i.e. $\mathcal{D}^* = \mathbb{R}[\partial_1, \dots, \partial_m]$, $\partial_i = \frac{\partial}{\partial c_i}$, $\text{Ann } \Psi = \{D \in \mathcal{D}^* \mid D\Psi = 0\}$). Let $\text{var}^j(\Psi) \subset \mathbb{R}[c_1, \dots, c_m]_{n-j}$ be the linear span of all partial derivatives of degree j of the polynomial Ψ . The linear map $(\mathcal{D}/\text{Ann } \Psi)_{2j} \rightarrow \text{var}^j \Psi$, $D \mapsto D\Psi$ is an isomorphism.

Let $\Delta = (\omega, \lambda)$ be a complete simplicial multi-fan on a finite set $M = [m]$ of rays in the space $V \cong \mathbb{R}^n$, where $\omega = \sum_{I \subset M, |I|=n} w(I)I$, $\lambda: M \rightarrow V$. Consider the simplicial complex K on M whose maximal simplices are all subsets $I \subset M$, $|I| = n$ such that $w(I) \neq 0$. K is called *the support* of the cycle ω ; it has dimension $n - 1$. If $J \in K$ is a simplex of any dimension, then we can define the projected multi-fan Δ_J as follows.

Consider the link of J in K : $\text{link}_K J = \{I \subset [m] \mid I \cap J = \emptyset, I \sqcup J \in K\}$, and let $M_J \subset M$ be the set of vertices of $\text{link}_K J$. Let V_J be the quotient of the vector space V by the subspace $\langle \lambda(i) \mid i \in J \rangle \cong \mathbb{R}^{|J|}$. Consider the simplicial cycle

$$\omega_J = \sum_{I \in \text{link}_K J, |I|=n-|J|} w(I \sqcup J)I \in Z_{n-1-|J|}(\text{link}_K J; \mathbb{R}).$$

The projected characteristic function $\lambda_J: M_J \rightarrow V$ is defined as the composition $M_J \subset M \xrightarrow{\lambda} V \twoheadrightarrow V_J$, where the last arrow is the natural projection. The multi-fan $\Delta_J = (\omega_J, \lambda_J)$ is called *the projected multi-fan* of Δ with respect to J .

Let $V_{\Delta} \in \mathbb{R}[c_1, \dots, c_m]_n$ be the volume polynomial of Δ . For a subset $J = \{i_1, \dots, i_j\} \subset [m]$ let ∂_J denote the differential operator $\partial_{i_1} \cdots \partial_{i_j}$. In [1, Lm.1] we proved that $\partial_J V_{\Delta}$ is zero whenever $J \notin K$; otherwise $\partial_J V_{\Delta}$ coincides with V_{Δ_J} , the volume polynomial of the projected multi-fan, up to epimorphic linear change of variables and up to constant factor. In particular, we have

$$(2.1) \quad \dim \text{var}^s \partial_J V_{\Delta} = \dim \text{var}^s V_{\Delta_J} \text{ for any } s = 0, \dots, n.$$

Let us recall the formula for the volume polynomial.

PROPOSITION 2.1 ([1]). *Let $\Delta = (\omega, \lambda)$ be as above and $v \in V$ be a generic vector (it will be called the polarization vector). Then*

$$(2.2) \quad V_\Delta(c_1, \dots, c_m) = \frac{1}{n!} \sum_{I=\{i_1, \dots, i_n\} \in K} \frac{w(I)}{|\det \lambda(I)| \prod_{j=1}^n \alpha_{I,j}} (\alpha_{I,1} c_{i_1} + \dots + \alpha_{I,n} c_{i_n})^n,$$

where $\alpha_{I,1}, \dots, \alpha_{I,n}$ are the coordinates of v in the basis $(\lambda(i_1), \dots, \lambda(i_n))$, $w(I)$ is the weight of the simplex I , and $\det \lambda(I)$ is the determinant of the matrix $(\lambda(i_1), \dots, \lambda(i_n))$.

The condition that v is generic means that all coefficients $\alpha_{I,i}$ are non-zero: this is an open condition in V .

REMARK 2.2. It can be seen from this formula that whenever the operator $A \in \text{GL}(V)$ acts on all values of characteristic function simultaneously, the volume polynomial does not change up to constant factor. Indeed, if we take Av as a polarization vector for the multi-fan $A\Delta = (\omega, A\lambda)$, all the coefficients $\alpha_{I,i}$ remain unchanged, and all the determinants $\det \lambda_I$ are multiplied by the same factor $\det A$. Therefore, $\mathcal{A}^*(A\Delta) \cong \mathcal{A}^*(\Delta)$.

We finish this section with a small remark, which will be used in the following.

REMARK 2.3. Let us fix a finite set $[m]$ and a function $\lambda: [m] \rightarrow V \cong \mathbb{R}^n$ which is general enough. All n -dimensional multi-fans on $[m]$ having λ as a characteristic function form a vector space (essentially, this is just a certain subspace of the space of all $(n-1)$ -cycles on $[m]$ vertices). Therefore, one can form sums and differences of multi-fans, provided that they have the same vertex sets and characteristic functions. Volume polynomial is additive with respect to this operation, which easily follows from its formula.

It may happen that the underlying cycle of a multi-fan does not contain some vertices from M . We call such vertices ghost vertices. The polynomial V_Δ does not actually depend on the variable c_i for any ghost vertex $i \in M$. In this case $\partial_i = 0$ in $\mathcal{A}^*(\Delta)$.

3. Values of characteristic function in smooth points

In this section we prove Theorem 1.3.

DEFINITION 3.1. A simplicial cycle $\omega \in Z_{n-1}(\Delta_{[m]}^{n-1}; \mathbb{R})$ is called *rigid* if the dimensions $d_j = \dim \mathcal{A}^{2j}(\Delta)$ of all multi-fans $\Delta = (\omega, \lambda)$ are independent of λ .

As was mentioned in the introduction, the fundamental cycle of any oriented homology manifold K is rigid since $d_j = h_j''(K)$. In particular, every homology sphere K is rigid and $d_j = h_j(K)$. If ω is rigid, we denote the dimension $\dim \mathcal{A}^{2j}$ by $d_j(\omega)$.

CONSTRUCTION 3.2. Let $\omega' = \sum_{|I'|=n'} w'(I') I'$, $\omega'' = \sum_{|I''|=n''} w''(I'') I''$ be two simplicial cycles on disjoint vertex sets M' , M'' . Define the join $\omega' * \omega''$ as a simplicial cycle on $M' \sqcup M''$:

$$\omega' * \omega'' = \sum_{\substack{I' \in M', |I'|=n' \\ I'' \in M'', |I''|=n''}} \omega'(I') \omega''(I'') I' \sqcup I'' \in Z_{n'+n''-1}(\Delta_{M' \sqcup M''}^{n'+n''-1}; \mathbb{R}).$$

Let us define the join of two multi-fans. Let $\Delta' = (\omega', \lambda')$ and $\Delta'' = (\omega'', \lambda'')$ be multi-fans in the spaces V' and V'' with the ray-sets M' and M'' respectively. Consider the multi-fan $\Delta' * \Delta'' = (\omega' * \omega'', \lambda)$, where $\lambda: M' \sqcup M'' \rightarrow V' \oplus V''$ is given by

$$\lambda(i) = \begin{cases} (\lambda'(i), 0), & \text{if } i \in M', \\ (0, \lambda''(i)), & \text{if } i \in M''. \end{cases}$$

There holds

$$V_{\Delta' * \Delta''} = V_{\Delta'} \cdot V_{\Delta''}.$$

This can be deduced either from the exact formula of the volume polynomial or geometrically, by noticing that every multi-polytope based on Δ is just the cartesian product of a multi-polytope based on Δ' and a multi-polytope based on Δ'' , so the volumes are multiplied. The polynomials $V_{\Delta'}$ and $V_{\Delta''}$ have distinct sets of variables, which implies

$$(3.1) \quad \mathcal{A}^*(\Delta' * \Delta'') \cong \mathcal{A}^*(\Delta') \otimes \mathcal{A}^*(\Delta'').$$

Therefore,

$$(3.2) \quad \text{Hilb}(\mathcal{A}^*(\Delta' * \Delta''); t) = \text{Hilb}(\mathcal{A}^*(\Delta'); t) \cdot \text{Hilb}(\mathcal{A}^*(\Delta''); t).$$

Let S^0 denote the simplicial complex consisting of two disjoint vertices x and y . By abuse of notation we use the same symbol S^0 to denote its underlying simplicial cycle lying in $Z_0(S^0; \mathbb{R})$. The join of a cycle ω with S^0 is called the suspension and is denoted by $\Sigma\omega$.

Δ is called a *suspension-shaped multi-fan* if its underlying simplicial cycle is isomorphic to $\Sigma\omega$ for some cycle ω . Note that the algebra of a suspension-shaped multi-fan contains two marked elements $\partial_x, \partial_y \in \mathcal{A}^*(\Delta)$ corresponding to the apices of the suspension. Since $\{x, y\}$ does not lie in the support of $\Sigma\omega$ we have the relation $\partial_x \partial_y = 0$ in $\mathcal{A}^*(\Delta)$. Let us consider two operators

$$\times \partial_x, \times \partial_y: \mathcal{A}^*(\Delta) \rightarrow \mathcal{A}^{*+2}(\Delta),$$

acting on the multi-fan algebra. There holds $\text{Im}(\times \partial_y) \subseteq \text{Ker}(\times \partial_x)$.

DEFINITION 3.3. A suspension-shaped multi-fan Δ is called *editable*, if

$$\text{Im}(\times \partial_y) = \text{Ker}(\times \partial_x).$$

Our next goal is to prove that suspensions over homology spheres are editable. Several technical lemmas are needed.

LEMMA 3.4. Let \mathcal{A}^* be a Poincare duality algebra of formal dimension $2n$ and $\mathcal{I} \subset \mathcal{A}^*$ be a graded ideal. Let $\text{Ann } \mathcal{I} := \{a \in \mathcal{A}^* \mid a\mathcal{I} = 0\}$ and let $\mathcal{I}^\perp = \bigoplus_j (\mathcal{I}^\perp)^{2j}$ denote the component-wise orthogonal complement:

$$(\mathcal{I}^\perp)^{2j} = \{a \in \mathcal{A}^{2j} \mid a\mathcal{I}^{2n-2j} = 0\}.$$

Then $\text{Ann } \mathcal{I} = \mathcal{I}^\perp$.

The proof is straightforward. We call ideals \mathcal{I} and $\text{Ann } \mathcal{I} = \mathcal{I}^\perp$ orthogonal. If Δ is a suspension-shaped multi-fan, then the ideal $\text{Im}(\times \partial_x)$ (which is just the principal ideal of $\mathcal{A}^*(\Delta)$ generated by ∂_x) is orthogonal to $\text{Ker}(\times \partial_x)$ by definition.

LEMMA 3.5. $\text{Hilb}(\text{Im}(\times \partial_x); t) = t^2 \text{Hilb}(\mathcal{A}^*(\Delta_x); t)$.

PROOF. Recall from §2 that there is an isomorphism $\mathcal{A}^{2j}(\Delta) \rightarrow \text{var}^j V_\Delta$ which sends D to DV_Δ . Therefore

$$\dim \text{Im}(\times \partial_x)_{2j} = \dim \{DV_\Delta \mid D \in \text{Im}(\times \partial_x)_{2j}\}.$$

The latter space may be identified with

$$\{D\partial_x V_\Delta \mid D \in \mathcal{D}_{2j-2}\} = \text{var}^{j-1}(\partial_x V_\Delta).$$

Equation (2.1) implies that $\dim \text{var}^{j-1}(\partial_x V_\Delta) = \dim \text{var}^{j-1} V_{\Delta_x} = \dim \mathcal{A}^{2j-2}(V_{\Delta_x})$. This finishes the proof. \square

LEMMA 3.6. *Suppose that a simplicial $(n-2)$ -cycle ω is rigid and its suspension $\Sigma\omega$ is rigid. Then any suspension-shaped multi-fan Δ on $\Sigma\omega$ is editable.*

PROOF. For any multi-fan Δ' based on S^0 we have $\text{Hilb}(\Delta'; t) = 1 + t^2$ (since S^0 is a sphere and its h-numbers are $(1, 1)$). Since ω , S^0 , and $\Sigma\omega$ are rigid, formula (3.2) implies

$$\text{Hilb}(\mathcal{A}^*(\Delta); t) = (1 + t^2) \sum_{j=0}^{n-1} d_j(\omega) t^{2j}.$$

By Lemma 3.5,

$$\text{Hilb}(\text{Im}(\times \partial_x); t) = t^2 \text{Hilb}(\mathcal{A}^*(\Delta_x); t).$$

Since Δ_x is a multi-fan based on ω , there holds $\text{Hilb}(\text{Im}(\times \partial_x); t) = t^2 \sum_{j=0}^{n-1} d_j(\omega) t^{2j}$. Similarly, $\text{Hilb}(\text{Im}(\times \partial_y); t) = t^2 \sum_{j=0}^{n-1} d_j(\omega) t^{2j}$. Using Lemma 3.4, we may find the dimensions of the orthogonal complement $\text{Ker}(\times \partial_x) = \text{Im}(\times \partial_x)^\perp$ in each degree:

$$\dim \text{Ker}(\times \partial_x)_{2j} = \dim \mathcal{A}^{2j}(\Delta) - \dim \text{Im}(\times \partial_x)_{2n-2j}.$$

This implies

$$\text{Hilb}(\text{Ker}(\times \partial_x); t) = t^2 \text{Hilb}(\mathcal{A}^*(\Delta_x); t) = \text{Hilb}(\text{Im}(\times \partial_y); t).$$

Thus $\text{Ker}(\times \partial_x) = \text{Im}(\times \partial_y)$. \square

COROLLARY 3.7. *Let K be a homology $(n-2)$ -sphere (or its underlying simplicial cycle). Then any suspension-shaped multi-fan Δ on ΣK is editable.*

PROOF. Suspension over a homology sphere is again a homology sphere. Thus both K and ΣK are rigid and Lemma 3.6 applies. \square

REMARK 3.8. The argument used in the proof of Lemma 3.6 shows that in general, if Δ is a suspension-shaped multi-fan with suspension points x, y , there holds

$$\text{Hilb}(\mathcal{A}^*(\Delta); t) \geq \text{Hilb}(\mathcal{A}^*(\Delta_x); t) + t^2 \text{Hilb}(\mathcal{A}^*(\Delta_y); t)$$

The next construction shows that suspension-shaped multi-fans arise naturally when we change the value of characteristic function at a single point.

CONSTRUCTION 3.9. Let $\Delta' = (\omega, \lambda'), \Delta'' = (\omega, \lambda'')$ be multi-fans based on the same simplicial cycle ω , and assume that $\lambda'(i) = \lambda''(i)$ for all $i \in [m]$ except $i = k$, where k is a fixed vertex. It is convenient to take two copies x, y of k and consider Δ' and Δ'' as the multi-fans on the set $M := ([m] \setminus \{k\}) \sqcup \{x, y\}$ having the same characteristic function λ :

$$\lambda(i) = \begin{cases} \lambda'(i) = \lambda''(i), & \text{if } i \in [m] \setminus \{k\}, \\ \lambda'(k), & \text{if } i = x, \\ \lambda''(k), & \text{if } i = y. \end{cases}$$

The underlying cycles of Δ' and Δ'' are isomorphic, but they are different as the elements of $Z_{n-1}(\Delta_M^{n-1}; \mathbb{R})$. The cycle ω' passes through x and has ghost vertex y , while the cycle ω'' passes through y and has ghost vertex x . Since Δ', Δ'' have the same characteristic function, their difference is well-defined:

$$T = \Delta'' - \Delta'.$$

It is easy to observe that T is a suspension-shaped multi-fan with suspension points x and y , whose underlying simplicial cycle has the form $\Sigma\omega_x = \Sigma\omega_y$ (recall that ω_x, ω_y are the projected simplicial cycles with respect to vertices x and y respectively).

THEOREM 3.10. *Let Δ', Δ'' be as above. If the projected $(n-2)$ -cycle ω_k is rigid and $T = \Delta'' - \Delta'$ is an editable suspension-shaped multi-fan, then $\text{Hilb}(\mathcal{A}^*(\Delta'); t) = \text{Hilb}(\mathcal{A}^*(\Delta''); t)$.*

PROOF. Consider the ring of differential operators $\mathcal{D}^* = \mathbb{R}[\partial_i \mid i \in M]$ and the principal ideals $(\partial_x), (\partial_y)$ in this ring. Note that $(\partial_y) \subset \text{Ann } V_{\Delta'}$ since y is a ghost vertex of Δ' , and similarly $(\partial_x) \subset \text{Ann } V_{\Delta''}$.

CLAIM 3.11. $\text{Ann } V_{\Delta'} + (\partial_x) = \text{Ann } V_{\Delta''} + (\partial_y)$

It is enough to show that $\text{Ann } V_{\Delta'} \subset V_{\Delta''} + (\partial_y)$ (the symmetry of the statement would imply $\text{Ann } V_{\Delta''} \subset V_{\Delta'} + (\partial_x)$).

First note that $V_{\Delta''} = V_{\Delta'} + V_T$. Hence $\partial_x V_{\Delta'} + \partial_x V_T = \partial_x V_{\Delta''} = 0$. Consider $D \in \text{Ann } V_{\Delta'}$. Then $\partial_x D V_T = -D \partial_x V_{\Delta'} = 0$. Therefore the class of D in the algebra $\mathcal{A}^*(T)$ lies in the kernel of $\times \partial_x: \mathcal{A}^*(T) \rightarrow \mathcal{A}^{*+2}(T)$. By assumption, T is editable, which implies $D \in \text{Ann } V_T + (\partial_y)$. Thus $D = D_1 + D_2$ where $D_1 \in \text{Ann } V_T$ and $D_2 \in (\partial_y)$.

Applying $D_1 = D - D_2$ to the polynomial $V_{\Delta''} = V_T + V_{\Delta'}$ we get

$$D_1 V_{\Delta''} = D_1 (V_T) + (D - D_2) V_{\Delta'} = 0$$

since $D \in \text{Ann } V_{\Delta'}$ by assumption and $D_2 \in (\partial_y) \subset \text{Ann } V_{\Delta'}$. Thus $D = D_1 + D_2$ where $D_1 \in \text{Ann } V_{\Delta''}$ and $D_2 \in (\partial_y)$ which proves the claim.

Now consider the diagram of inclusions of ideals:

$$(3.3) \quad \begin{array}{ccccc} \text{Ann } V_{\Delta'} & \hookrightarrow & \text{Ann } V_{\Delta'} + (\partial_x) = \text{Ann } V_{\Delta''} + (\partial_y) & \longleftarrow & \text{Ann } V_{\Delta''} \\ \uparrow & & \uparrow & & \uparrow \\ \text{Ann } V_{\Delta'} \cap (\partial_x) & \hookrightarrow & (\partial_x, \partial_y) & \longleftarrow & \text{Ann } V_{\Delta''} \cap (\partial_y) \end{array}$$

CLAIM 3.12. $\text{Hilb}(\text{Ann } V_{\Delta'} \cap (\partial_x); t) = \text{Hilb}(\text{Ann } V_{\Delta''} \cap (\partial_y); t)$.

We have $\text{Ann } V_{\Delta'} \cap (\partial_x) = \{\partial_x D \in \mathcal{D} \mid \partial_x D V_{\Delta'} = 0\}$. Up to the shift of grading this vector space coincides with $\text{Ann}(\partial_x V_{\Delta'})$. There holds $\text{Hilb}(\text{Ann}(\partial_x V_{\Delta'}); t) = \text{Hilb}(\mathcal{D}; t) - \text{Hilb} \mathcal{A}^*(\partial_x V_{\Delta'})$. Note that $\dim \mathcal{A}^{2j}(\partial_x V_{\Delta'}) = \dim \text{var}^j(\partial_x V_{\Delta'})$ and the latter space has the same dimension as $\dim \text{var}^j(V_{\Delta'_x}) = \dim \mathcal{A}^{2j}(\Delta'_x)$ according to equation (2.1). Both projected multi-fans Δ' and Δ'' are based on the same projected cycle ω_k . By the assumption of the theorem, ω_k is rigid, thus $\dim \mathcal{A}^{2j}(\Delta'_x) = \dim \mathcal{A}^{2j}(\Delta''_y)$. This proves the claim.

Finally, the diagram (3.3) and the claim imply $\text{Hilb}(\text{Ann } V_{\Delta'}; t) = \text{Hilb}(\text{Ann } V_{\Delta''}; t)$ which proves the theorem. \square

Corollary 3.7 and Theorem 3.10 together imply Theorem 1.3.

4. Singular examples

PROPOSITION 4.1. *There exist two complete simplicial multi-fans Δ_1, Δ_2 having the same underlying simplicial cycle but different dimension vectors.*

PROOF. Let $m = 6$ and $n = 2$. Consider the oriented graph Γ depicted in Fig.1. Let $\omega_\Gamma \in C_1(\Delta_{[6]}; \mathbb{Z})$ be the simplicial chain which is the sum of all oriented edges of Γ with weights 1. Since Γ is eulerian, ω_Γ is a cycle.

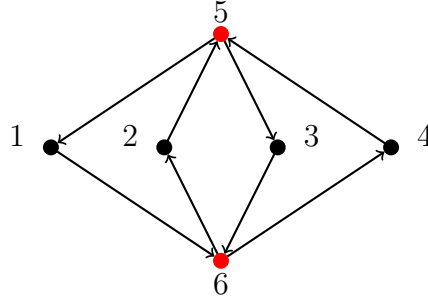


FIGURE 1. Graph Γ

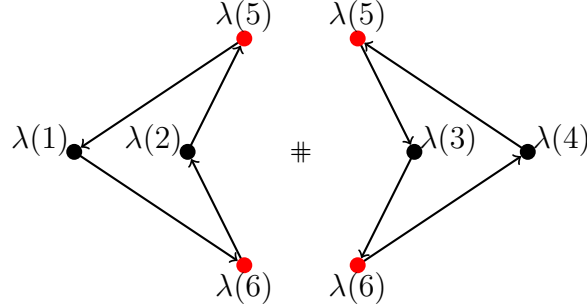
Let us define two complete simplicial multi-fans Δ_1, Δ_2 with underlying cycle ω_Γ . To do this, we need to specify the values of characteristic functions $\lambda_{1,2}: [6] \rightarrow \mathbb{R}^2$.

(1) Define the function λ_1 so that its values in the vertices 1, 2, 3, 4 lie on x -axis and the values in vertices 5, 6 lie in y -axis. For example, take $\lambda_1(1) = \lambda_1(2) = \lambda_1(3) = \lambda_1(4) = e_1$, $\lambda_1(5) = \lambda_1(6) = e_2$, where e_1, e_2 is the basis of \mathbb{R}^2 . In this case the multi-fan $\Delta_1 = (w_\Gamma, \lambda_1)$ is the join of two 1-dimensional multi-fans Δ' and Δ'' depicted below:

$$\begin{array}{ccccccc}
 & & & & & & e_2 \bullet + 1 \\
 & e_1 & e_1 & e_1 & e_1 & & \\
 \bullet & \bullet & \bullet & \bullet & & * & \\
 -1 & +1 & -1 & +1 & & & \\
 & & & & & & e_2 \bullet - 1
 \end{array}$$

Equation (3.1) tells that $\mathcal{A}^*(\Delta_1) = \mathcal{A}^*(\Delta') \otimes \mathcal{A}^*(\Delta'')$. Algebra of any 1-dimensional multi-fan has Hilbert function $1 + t^2$ due to Poincare duality. Finally, $\text{Hilb}(\mathcal{A}^*(\Delta_1); t) = (1 + t^2)(1 + t^2) = 1 + 2t^2 + t^4$.

(2) Let us define the function $\lambda_2: [6] \rightarrow \mathbb{R}^2$. Set the values $\lambda_2(1), \lambda_2(2), \lambda_2(3), \lambda_2(4)$ arbitrarily (for example we may set them equal to e_1). Now choose $\lambda_2(5)$ and $\lambda_2(6)$ so that they are linearly independent. Then $\Delta_2 = (w_\Gamma, \lambda_2)$ may be represented as a connected sum of 2-dimensional multi-fans $\dot{\Delta}, \ddot{\Delta}$ depicted below:



(The definition of connected sum is given in [1]. There we do not require that the set along which the connected sum is taken is a simplex: it is only required that the values of characteristic function on this set are linearly independent.) By [1, Prop.11.3] we have $\text{Hilb}(\Delta_2; t) = \text{Hilb}(\dot{\Delta}; t) + \text{Hilb}(\ddot{\Delta}; t) - (1 + t^4)$. Multi-fans $\dot{\Delta}, \ddot{\Delta}$ are supported on spheres, therefore dimensions of their algebras are the h -vectors. These are $(1, 2, 1)$ in both cases. Thus $\text{Hilb}(\Delta_2; t) = 1 + 4t^2 + t^4$. \square

REMARK 4.2. The vertices 1, 2, 3, 4 of Γ are smooth vertices. Hence the Hilbert function of $\mathcal{A}^*(\Delta)$ does not depend on the values of λ in these vertices by Theorem 1.3. Proposition 4.1 implies the following alternative: the dimensions-vector of a multi-fan on Γ is either $(1, 2, 1)$ (if the values $\lambda(5), \lambda(6)$ are collinear) or $(1, 4, 1)$ (if $\lambda(5), \lambda(6)$ are linearly independent).

PROPOSITION 4.3. *There exist two complete simplicial multi-fans Δ_1, Δ_2 which are supported on the same pseudomanifold but have different dimensions of their multi-fan algebras.*

PROOF. Let L be the minimal triangulation of a 2-torus shown on Fig.2. Using general formulas for h'' -numbers, one can show that h'' -numbers of L are $(1, 4, 4, 1)$. Let Δ' be any multi-fan supported on L . Since L is a manifold, we have $\text{Hilb}(\mathcal{A}^*(\Delta'); t) = 1 + 4t^2 + 4t^4 + t^6$.

Consider the suspension $K = \Sigma L$. We claim that there exist two multi-fans supported on K with different d-vectors of multi-fan algebras

(1) Let Δ'' be any multi-fan supported on S^0 . Then we have $\text{Hilb}(\mathcal{A}^*(\Delta''); t) = 1 + t^2$. Therefore,

$$\text{Hilb}(\mathcal{A}^*(\Delta' * \Delta''); t) = (1 + 4t^2 + 4t^4 + t^6)(1 + t^2),$$

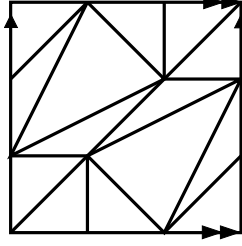


FIGURE 2. Minimal triangulation of a torus

so the dimension vector of multi-fan $\Delta' * \Delta''$ supported on K is $(1,5,8,5,1)$. This multi-fan has the property that the values of its characteristic function in suspension points are collinear.

(2) Now we take another multi-fan supported on K and set the values of characteristic function in suspension points to be non-collinear. In this case we do not have an easy algorithm to compute the dimensions by hand. However, the procedure can be easily implemented in GAP. The following program computes the volume polynomial and the dimension vector:

```

torus:=SCLib.Load(4);
K:=SCSuspension(genus4);
SCRelabelStandard(K);

sign:=SCOrientation(K);
n:=SCDim(K)+1;
m:=Length(SCVerticesEx(K));
MaxSimp:=SCFacets(K);

RandomVector:=function() local X; X:=[];
    for j in [1..n] do
        Add(X,Random([-15..15]));
    od;
    return X;
end;

ArrayRV:=[]; for i in [1..m+1] do
    Add(ArrayRV,RandomVector());
od;

CharFunc:=ArrayRV{[1..m]};
Polar:=ArrayRV[m+1];

#CharFunc[m]:=CharFunc[m-1];
    
```

```

VolPol:=0;

Clist:=[]; for i in [1..m] do
  Add(Clist,X(Rationals,Concatenation("c",ViewString(i))));
od;

r:=1;
for I in MaxSimp do
  Matr:=[];
  for i in I do
    Add(Matr,CharFunc[i]);
  od;
  alpha:=Polar*Matr^-1;
  Form:=0;
  j:=1;
  for i in I do
    Form:=Form+\alpha[j]*Clist[i];
    j:=j+1;
  od;
  D:=DeterminantMat(Matr);
  Form:=sign[r]*Form^n/Product(alpha)/D;
  r:=r+1;
  VolPol:=VolPol+Form;
od; VolPol:=VolPol/Factorial(n);

Display(VolPol);

dims:=[]; for j in [1..n-1] do
  ListOfDerivs:=[];
  for J in UnorderedTuples([1..m],j) do
    Deriv:=VolPol;
    for a in J do
      Deriv:=Derivative(Deriv,Clist[a]);
    od;
    Add(ListOfDerivs, Deriv);
  od;
  Add(dims,Dimension(VectorSpace(Rationals,ListOfDerivs)));
od;
Add(dims,1);

Display(dims);

```

The program outputs the d-vector $(1, 5, 12, 5, 1)$ whenever the values of λ in suspension points (last two values of the list) are non-collinear. \square

In fact, there are exactly two alternatives for the dimension vector of the suspension over a torus.

PROPOSITION 4.4. *Let N be a triangulation of a 2-torus having $m-2$ vertices, and $K = \Sigma N$ its suspension with additional suspension points x, y . For a multi-fan $\Delta = ([K], \lambda)$ supported on the pseudomanifold K there are two alternatives:*

- (1) *d-vector equals $(1, m-4, 2m-10, m-4, 1)$ if the vectors $\lambda(x), \lambda(y)$ are collinear;*
- (2) *d-vector equals $(1, m-4, 2m-6, m-4, 1)$ if the vectors $\lambda(x), \lambda(y)$ are non-collinear.*

PROOF. At first note that all vertices of N , that is $M \setminus \{x, y\}$, are smooth in the pseudomanifold K . Therefore, d-vector of a multi-fan does not depend on the values of characteristic function in these vertices by Theorem 1.3.

(1) Let M denote the set of vertices of K , $|M| = m$. Let $\lambda(x), \lambda(y) \in V \cong \mathbb{R}^4$ be collinear. We may assume that all values $\{\lambda(i) \mid i \in M \setminus \{x, y\}\}$ lie in the 3-plane transversal to $\langle \lambda(x) \rangle$. Then Δ is just the join of two multi-fans: one supported on N and another supported on S^0 . Formula (3.2) implies

$$\text{Hilb}(\mathcal{A}^*(\Delta); t) = (h_0''(N) + h_1''(N)t^2 + h_2''(N)t^4 + h_3''(N)t^6)(1 + t^2).$$

h'' -numbers of any 2-surface are easy to compute: they are symmetric, $h_0'' = 1$, and h_1'' equals the number of vertices minus 3. Therefore,

$$\text{Hilb}(\mathcal{A}^*(\Delta); t) = (1 + (m-5)t^2 + (m-5)t^4 + t^6)(1 + t^2) = 1 + (m-4)t^2 + (2m-10)t^4 + (m-4)t^6 + t^8.$$

This proves the first case.

(2) Now suppose that $\lambda(x), \lambda(y)$ are non-collinear. At first, we claim, that proposition holds for the minimal triangulation L of a torus.

Indeed, the statement holds for some particular choice of characteristic function as was shown by direct calculation (see the proof of Proposition 4.3). In this case we have $m = 9$ and d-vector is $(1, 5, 12, 5, 1)$. However any two non-collinear pairs of vectors (values in suspension points) may be translated into each other by an element $A \in \text{GL}(V)$. Therefore the d-vector is $(1, 5, 12, 5, 1)$ for any characteristic function on ΣL according to remark 2.2.

Let us prove the statement for an arbitrary triangulation of a torus. Any triangulation N is connected to the minimal one by a sequence of bistellar moves according to Pachner's theorem [11]. Therefore the corresponding sequence of "suspended bistellar moves" joins $K = \Sigma N$ with ΣL . There are 3 types of bistellar moves in dimension 2, shown on the top of Fig.3. Suspended bistellar moves are shown below.

Each suspended move is decomposed as a sequence of 3-dimensional bistellar moves. In [1, Thm.10] we proved that bistellar moves (otherwise called flips) performed on a multi-fan have a very predictable effect on the dimension vector of its algebra: the d-vector changes in exactly the same way as the h-vector does. Let us make all the computations.

The suspended move $\Sigma(0, 2)$ is the same as adding new vertex d in the tetrahedron $\{a, b, c, y\}$ (this is $(0, 3)$ -move) followed by the $(1, 2)$ -move applied to adjacent tetrahedra $\{a, b, c, d\}$ and $\{a, b, c, x\}$. The $(0, 3)$ -move adds $(0, 1, 1, 1, 0)$ to the dimension vector and

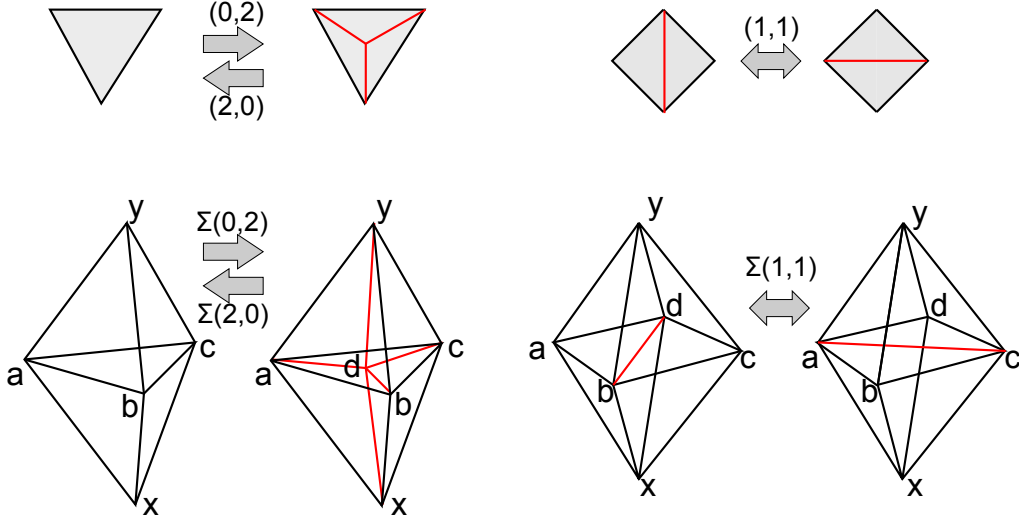


FIGURE 3. Bistellar moves and their suspensions

(1, 2)-move adds $(0, 0, 1, 0, 0)$. Therefore $\Sigma(0, 2)$ increases $d_1 = d_3$ by 1 and d_2 by 2. The inverse move $\Sigma(2, 0)$ decreases d-vector in the same way.

The suspended move $\Sigma(1, 1)$ is equivalent to the application of (1, 2)-move to adjacent tetrahedra $\{a, b, d, y\}$ and $\{b, c, d, y\}$ followed by the application of (2, 1)-move to tetrahedra $\{a, b, c, d\}, \{a, b, d, x\}, \{b, d, c, x\}$. First move adds $(0, 0, 1, 0, 0)$ to the d-vector, and the second subtracts the same value. So far, under the suspended (1, 1)-move d-vector remains unchanged.

In all three cases d-vector changes in the same way, as the expression $(1, m - 4, 2m - 6, m - 4, 1)$. Since d-vector is equal to this expression for the minimal triangulation of a torus, the same holds for any triangulation of a 2-torus. \square

REMARK 4.5. Propositions 4.3, 4.4 show that suspension K over a 2-torus is not rigid. A suspension-shaped multi-fan supported on K is editable if and only if the values of its characteristic function in the suspension points are collinear.

Propositions show that multi-fans supported on a suspension of a fixed triangulation of a torus may have two different values of d-vector. The technique used in the proof shows that d-vectors of multi-fans supported on suspended orientable surfaces of any genus $g \geq 2$ can take no more than 2 values, depending on whether the values of λ in suspension points are collinear or not. The results of calculations performed in GAP support the following claim.

CLAIM 4.6. *Let N be a triangulated surface of genus g with $m - 2$ vertices, and $K = \Sigma N$ its suspension with additional suspension points x, y . For a multi-fan $\Delta = ([K], \lambda)$ supported on the pseudomanifold K there are two alternatives:*

- (1) *d-vector equals $(1, m - 4, 2m - 10, m - 4, 1)$ if the vectors $\lambda(x), \lambda(y)$ are collinear;*

- (2) *d*-vector equals $(1, m - 4, 2m - 10 + 4g, m - 4, 1)$ if the vectors $\lambda(x), \lambda(y)$ are non-collinear.

So far, the gap between possible values of *d*-vector depends on the links of singular points. The claim was computed on examples with $g \leq 10$ however we cannot explain this result mathematically.

5. 3-dimensional pseudomanifolds

Let X be any triangulated 3-dimensional closed oriented pseudomanifold with isolated singularities. By this we mean that X is a pure 3-dimensional simplicial complex such that links of all its vertices are orientable surfaces and X has a fundamental cycle. We will also assume that singular points are not connected by edges in the triangulation.

Consider the following number

$$r(X) = \text{number of distinct d-vectors of multi-fans supported on } X.$$

PROPOSITION 5.1. *$r(X)$ is a topological invariant, that is $X_1 \cong X_2$ implies $r(X_1) = r(X_2)$.*

PROOF. Any two triangulations of a given topological pseudomanifold with isolated singularities are connected by a sequence of bistellar moves performed outside singularities, as was shown in [2, Th.4.6]. However, each bistellar move have the same effect on all *d*-vectors, so the number of possible *d*-vectors coincides for all triangulations. \square

EXAMPLE 5.2. As was proved earlier, $r(X) = 1$ for all manifolds. If the pseudomanifold X have only one singular point, we still have $r(X) = 1$ since the value of λ in the singular point (the only value that matters according to Theorem 1.3) can be made arbitrary by a linear transform of the ambient space. If X is a connected sum of several pseudomanifolds X_i with $r(X_i) = 1$, then $r(X)$ also equals 1. Indeed, in [1] we showed that $d_j(\Delta_1 \# \Delta_2) = d_j(\Delta_1) + d_j(\Delta_2)$ for $j \neq 0, n$, therefore there is only one possibility for the *d*-vector of connected sum, whenever this is true for the summands. Hence there exist pseudomanifolds X with any number of singular points having $r(X) = 1$.

However, $r(X) = 2$ for the suspended torus (and conjecturally for all suspended surfaces by Claim 4.6). By taking connected sums of suspended tori, we can construct pseudomanifolds X with arbitrarily large $r(X)$.

EXAMPLE 5.3. Consider two 3-pseudomanifolds: X_1 is the suspended 2-torus and X_2 is the connected sum of two copies of the space Y , where Y is the quotient of the solid torus by its boundary. We have $r(X_1) = 2$ by Proposition 4.4 and $r(X_2) = 1$ by the previous example, since the summand Y have only one singularity. The spaces X_1 and X_2 are different, but this difference is not easy to see. The cohomology rings are isomorphic: in both cases cohomology is torsion-free, and the Betti numbers are $(1, 0, 2, 1)$. The multiplication is trivial by dimensional reasons. Also both spaces have exactly two singular points with toric links. The difference may be seen by cutting singular points and noticing that the first space becomes S^3 minus Hopf link, while the second space becomes S^3 minus two unlinked circles. It is interesting that invariant r can sense such knot-theoretical distinctions.

The invariant $r(X)$ somehow measures the complexity of spatial relationships between singular points. It would be interesting to describe this number in a more formal and computable way or at least find out when $r(X) = 1$. The next question is motivated by example 5.3.

PROBLEM 5.4. *Let $l: \bigsqcup_{\alpha} S_{\alpha}^1 \hookrightarrow S^3$ be a link (in a knot-theoretical meaning) and X be a pseudomanifold obtained by collapsing each component of l to a point. Is it true that $r(X) = 1$ if and only if each two circles of the link are unlinked?*

Example 5.3 shows that pairwise linking numbers affect $r(X)$. However, the question remains: is the linking number the only thing that matters? We wanted to test any Brunnian link.

PROPOSITION 5.5. *Let $l: \bigsqcup_{\alpha=1,2,3} S_{\alpha}^1 \hookrightarrow S^3$ be the Borromean rings and X be a pseudomanifold obtained by collapsing each component of the link to a point. Then $r(X) = 1$.*

PROOF. X can be triangulated as follows. At first note that X is obtained by cutting the neighborhoods of Borromean rings from S^3 and inserting the cone over each boundary component.

Therefore we specialize three tori in S^3 linked together like Borromean rings (see left part of Fig.5), triangulate the remaining space, and put a cone over each torus. Consider tori shown in Fig.4 (each quadrangular face should be further triangulated). Some parts of tori should be identified as the labels show. The space remaining after deletion of solid tori consists of the inner cube and the outer space (see the lower part of Fig.4). The first one can be triangulated by taking a cone with vertex in the origin, and the second can be triangulated by taking a cone with vertex at infinity.

With figure 4 in hand the space X can be easily encoded in GAP. Now we need to specify the values of characteristic function. Only the values in three singular points may affect the result. In our implementation singular vertices (which are the apices of the cones over tori) are 29, 30, 31. The symmetry group of X acts transitively on the set of singular vertices, therefore, up to linear transformation of the ambient vector space V we have the following possibilities:

- (1) $\lambda(29), \lambda(30)$ and $\lambda(31)$ are linearly independent. Without loss of generality, $\lambda(29) = (1, 0, 0, 0)$, $\lambda(30) = (0, 1, 0, 0)$, $\lambda(31) = (0, 0, 1, 0)$.
- (2) $\lambda(29), \lambda(30)$ and $\lambda(31)$ lie in one 2-space, but any two of them are non-collinear. W.l.o.g., $\lambda(29) = (1, 0, 0, 0)$, $\lambda(30) = (0, 1, 0, 0)$, $\lambda(31) = (1, 1, 0, 0)$.
- (3) $\lambda(29)$ is collinear to $\lambda(30)$ and non-collinear to $\lambda(31)$. W.l.o.g., $\lambda(29) = (1, 0, 0, 0)$, $\lambda(30) = (1, 0, 0, 0)$, $\lambda(31) = (0, 1, 0, 0)$.
- (4) $\lambda(A), \lambda(B)$, and $\lambda(C)$ are collinear. W.l.o.g., $\lambda(29) = \lambda(30) = \lambda(31) = (1, 0, 0, 0)$.

All four cases are checked in GAP, and the resulting d-vector is $(1, 27, 100, 27, 1)$ in all four cases. Therefore, for collapsed borromean rings we have $r(X) = 1$. \square

REMARK 5.6. We computed another example shown on the right part of Fig.5. After collapsing each component of this link we obtain a pseudomanifold with 3 singular points. We considered a special triangulation of this space, constructed similarly to the

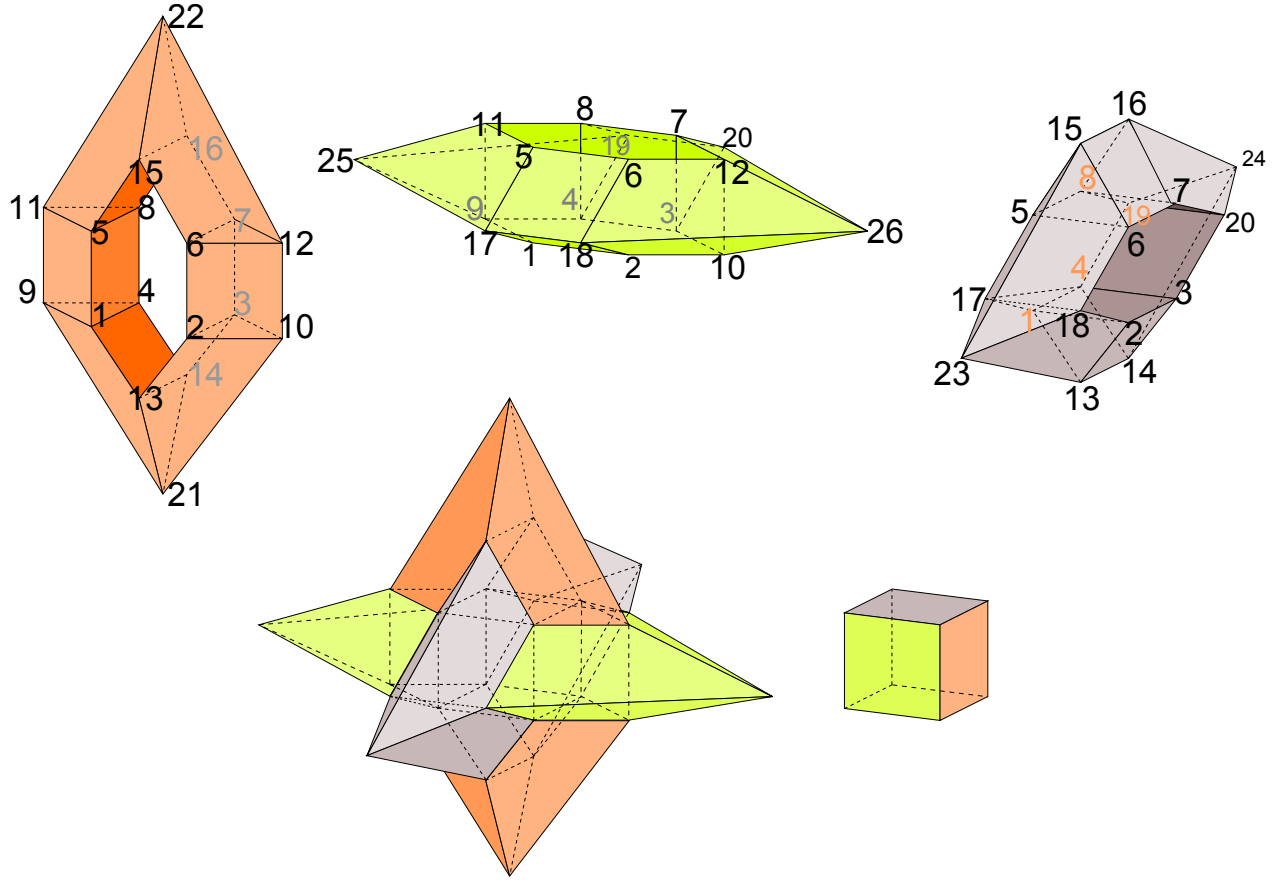


FIGURE 4. Parts of construction of collapsed Borromean rings



FIGURE 5. Two links

case of Borromean rings. This triangulation has 18 vertices with singular vertices labeled 16 (corresponds to middle circle), 17, and 18. Calculations had shown that there are 3 alternatives:

- (1) if $\lambda(16)$, $\lambda(17)$, $\lambda(18)$ are collinear, then d-vector is $(1, 14, 34, 14, 1)$;

- (2) if $\lambda(16), \lambda(17), \lambda(18)$ span 2-dimensional space, then d-vector is $(1, 14, 38, 14, 1)$;
- (3) if $\lambda(16), \lambda(17), \lambda(18)$ are linearly independent, then d-vector is $(1, 14, 40, 14, 1)$.

Surprisingly, the relation between the values of characteristic function at singular points corresponding to unlinked circles (vertices labeled by 17, 18) affect the answer. Indeed, when $\lambda(17), \lambda(18), \lambda(16)$ are all in general position, the answer differs from the case when $\lambda(17) = \lambda(18)$, and $\lambda(16)$ is in general position. This is another strange phenomenon which should be explained.

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